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Application of Resolvent CCR Algebras to Statistical Mechanics of Bosons on Lattices

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1 Introduction

In this note, we report application of the resolvent CCR algebras to statistical mechanics of Bosons on lattices. One of our motivation is originated from results on the quantum Ising model in a transversal magnetic field.

Let us recall known results of the quantum Ising model. First we consider the case of the model on one dimensional integer lattice . Let $\mathfrak{A}$ be the UFH C^*-algebra which is an infinite tensor product of the algebra of 2 by 2 matrices where each component of the tensor is specified with a site in the integer lattice. We denote the sub-algebra of observables localized in Λ by $\mathfrak{A}_\Lambda$ and we set $\mathfrak{A}_{loc} = \cup_{\Lambda\subset \mathbf{Z},|\Lambda|<\infty}\mathfrak{A}_\Lambda$. For each natural number N the finite volume Hamiltomian H_N of the quantum Ising model on $[-N, N]$ is defined by the following equation:

$$H_N = -\sum_{j=-N}^{N-1} \sigma_z^{(j)}\sigma_z^{(j+1)} + \lambda \sum_{j=-N}^{N} \sigma_x^{(j)}$$

where $\sigma_{x,z}^{(j)}$ are Pauli spin matrices on the site j and λ is a real parameter. The limit

$$\alpha_t(Q) = \lim_{N\to\infty} e^{itH_N}Qe^{-itH_N}$$

exists in the norm topology and the Heisenberg time evolution α_t gives rise to a C^*- dynamical system of $\mathfrak{A}$.

A state ω of $\mathfrak{A}$ is a β-KMS state if the time-dependent correlation function defined by $F_{Q_1,Q_2}(t) \equiv \omega_\beta(\alpha_t(Q_1)Q_2)$ satisfy the following KMS condition:

$$F_{Q_1,Q_2}(t) = F_{Q_2,Q_1}(t+i\beta)$$

for any Q_1 and Q_2 which are entire analytic for α_t. Note that Q is entire analytic if $\alpha_t(Q)$ as a function of t has an analytic extension to the whole complex plain. The set of analytic elements is dense in $\mathfrak{A}$. As the thermal equilibrium state at the inverse temperature β satisfy this KMS condition, it is natural to regard β-KMS states as the equilibrium states for quantum systems with an infinite degree of freedom.

In the same spirit, we can introduce infinite volume ground states, namely, a state ω of $\mathfrak{A}$ is a ground state if the inequality

$$\lim_{N\to\infty} \omega(Q^*[H_N, Q]) \geq 0 \quad (Q \in \mathfrak{A}_{loc})$$

holds.The zero temperature limit of KMS states is a ground state defined in this manner.

In 1975, H.Araki proved uniqueness of β-KMS state for any one-dimensional quantum spin system with any short range interaction Hamiltonian. The ground state of the finite volume quantum Ising model is unique, and for the infinite volume ground states of the quantum Ising model, the following results are known.

Theorem 1.1 *(H. Araki ,Taku Matsui 1982CMP)*

(i) there exist precisely two infinite volume pure ground states if $|\lambda| < 1$

(ii) the ∞ *volume ground state is unique if if* $|\lambda| \geq 1$

For higher dimensional lattices $\mathbf{Z}^d$ ($2 \leq d$) the KMS state of quantum Ising model is unique at high temperature and at least two extremal low temperature KMS states exist. There exist two pure ground states when the transversal field is weak.

The problem we consider here is a Bosonic counterpart of quantum Ising models and in our opinion, a natural candidate is anharmonic crystals on $\mathbf{Z}^d$. The Hamiltonian of the quantum anharmonic crystal is written in the following form.

$$H = \sum_{k\in\mathbf{Z}^d} \{p_k^2 + V(x_k)\} + \sum_{k,l:|k-l|=1} \varphi(x_k - x_l)$$

where $V(x)$ is a double well potential.

$$\lim_{x\to\infty} V(x) = \infty$$

with two local minima and φ represents interaction between adjacent particles.

One of mathematical difficulty for interacting Bose systems is to define the Heisenberg time evolution of quantum observables. In suitable setting, it is possible to show that the Heisenberg time evolution of the quantum anharmonic crystal does not exist as one-parameter group of automorphisms of the Weyl CCR algebra. Due to this fact , we employ the resolvent CCR algebra introduced by H.Grundling and D.Buchholz for their study of supersymmetric QFT.

Formally , the resolvent CCR algebra is generated by resolvent of linear combination of x_k and p_k. For example, in a quantum system with one degree of freedom, set

$$R_\lambda(s,t) = [\lambda i1 + (sx+tp)]^{-1} \quad \lambda, s, u \in \mathbf{R}$$

and the resolvent CCR algebra is the universal C^*-algebra generated by $R_\lambda(s,t)$ and a unit.

More generally, let (V,σ) be a real symplectic space. R_{BG} is the universal C^*-algebra generated by a unit and $\{R_\lambda(v)|\ v \in V\}$ satisfying relations For $\lambda,\ \nu \neq 0$, $f,g \in V$

$$R_\lambda(0) = -\frac{i}{\lambda}, \quad R_\lambda(f)^* = R_{-\lambda}(f), \quad \nu R_{\nu\lambda}(\nu f) = R_\lambda(f)$$

$$R_\lambda(f) - R_\nu(f) = i(\nu - \lambda)R_\lambda(f)R_\nu(f)$$

$$[R_\lambda(f), R_\nu(g)] = i\sigma(f,g)R_\lambda(f)R_\nu(g)^2 R_\lambda(f)$$

$$R_\lambda(f)R_\nu(g) = R_{\lambda+\nu}(f+g)\{R_\lambda(f) + R_\nu(g) + \sigma(f,g)R_\lambda(f)^2 R_\nu(g)\}$$

Due to these relations, it is easy to see that there exists a trivial representation π of R_{BG} such that $\pi(R_\lambda(f)) = 0$.For representations and states of R_{BG} we require that field operators satisfying CCR(canonical commutation relations) can be reconstructed from $R_\lambda(f)$.

Definition 1.2 *(H.Grundling and D.Buchholz)*
A representation π is regular if $\pi(R_\lambda(f))$ is a resolvent of a closed operator for any $f \in V$,

Regularity of a represetation is equivalent to the condition $ker(R_\lambda(f)) = \{0\}$ for any $f \in V$ H.Grundling and D.Buchholz have shown that the CCR algebra can be reproduced for any regular representation of R_{BG}. They have shown the standard Fock representation is a regular faithful representation of R_{BG} as well.

2 Weakly Coupled Anharmonic Crystal

Now we consider the resolvent CCR algebra R_{BG} associated with weakly coupled anharmonic crystals. On each lattice site k in $\mathbf{Z}^d$, we have $L_2(\mathbf{R}^d)$. we set $V = \mathbf{R}^{2\infty}$ which is a infinite direct sum of two dimensional symplectic space. Here we assume that all but finite components of the summand vanish.

We introduce the finite volume Hamiltonian of the weakly coupled anharmonic crystal as follows:

$$H_\Lambda = \sum_{k\in\Lambda} \{p_k^2 + w^2 x_k^2 + V(x_k)\} + \sum_{k,l\in\Lambda,|k-l|=1} \varphi(x_k - x_l)$$

Here we assume both $V(x)$ and $\varphi(x)$ are continuous functions, vanishing at infinity on $\mathbf{R}^d$.

The word "weakly coupled" is employed here as we presume $\varphi(x)$ vanishes at infinity. We can show that

$$e^{itH_\Lambda} Q e^{-itH_\Lambda} \quad Q \in R_{BG}$$

is a well-defined automorphism of R_{BG} , but is not norm continuous in t. Thanks to Lieb-Robinson Bounds on the Fock representation, it is possible to prove existence of the infinite volume dynamics (the Heisenberg time evolution as automorphisms R_{BG})

Theorem 2.3 *The limit*

$$\lim_{\Lambda \to \mathbf{Z}^d} e^{itH_\Lambda} Q e^{-itH_\Lambda} = \alpha_t(Q)$$

exists in the operator norm topology of the resolvent algebra, and $\pi_F(\alpha_t(Q))$ is weakly continuous in t for the Fock representation π_F.

As a function of t, $\alpha_t(Q)$ is not continuous in the norm topology of R_{BG}, nevertheless, we can define the β-KMS state.

We say a state ω of R_{BG} is a β-KMS state if the following three conditions are valid:

(i) The time dependent correlation function $F_{Q_1,Q_2}(t) \equiv \omega_\beta\left(\alpha_t(Q_1)Q_2\right)$ is continuous for any $t \in \mathbf{R}$

(ii) For any Q_1 and Q_2 in R_{BG} there exists a complex function $G_{Q_1,Q_2}(z)$ which is holomorphic in the strip $\{z|0 < Imz < \beta\}$, and is bounded ,continuous on the boundary of $\{z|0 < Imz < \beta\}$ such that $G_{Q_1,Q_2}(t) = F_{Q_1,Q_2}(t)$.

(iii) $G_{Q_1,Q_2}(t) = G_{Q_2,Q_1}(t)(t + i\beta)$

Theorem 2.4 *For any $\beta > 0$, there exists a β-KMSstate ω_β of the resolvent algebra such that for any finite subset $\Lambda \subset \mathbf{Z}^d$ the restriction of ω_β to observables localized in Λ is normal to the Fock representation.*

This above normality is called locally normal to the Fock representation. If a KMS state ω is locally normal to the Fock representation, we say ω is a regular KMS state.

Remark 2.5 *The following remarks are in order.*

(i) *D.Buchholz considered a similar model in a slightly different setting. He restricted α_t to a sub-algebra for which α_t is continuous in t. The disadvantage of the approach is in the point that the sub-algebra does not contain functions of position operators and momentum operators.*

(ii) *Non-regular KMS states exist , though , we cannot construct field operators in the GNS representation of non-regular KMS states, and we regard non-regular KMS states unphysical.*

By use of H.Araki's argument of relative entropy for quantum spin chains, and results of relative entropy of (not necessarily C^*-) *algebras due to A.Uhlmann, we can establish absence of phase Transition in one dimensional systems.

Theorem 2.6 *If the dimension of the lattice is one, the regular KMS state of the weakly coupled anharmonic crystal is unique.*

Our weakly coupled anharmonic crystal is the first example of interacting quantum lattice models with unbounded spin in which uniqueness of KMS states established.

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